# MATHS REVIEW

2018/2019

STOWE

## Gaussian Group

2018 – 2019

## Predator-Prey Modelling

The Logistical Curve and The Lotka-Volterra Equations

## Taylor Series

# The Stowe Maths Review

The Stowe Maths Review is a magazine that gives an insight into maths at Stowe School.

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**Special thanks to:**

All members of the Gaussian Group.

Mr Karakus for his support.

All the maths teachers in the department.

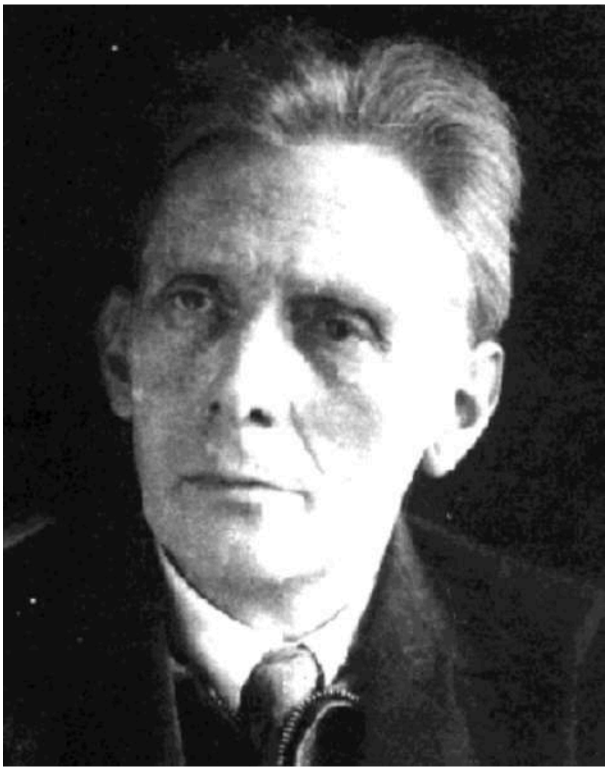
Tori Roddy for helping us to publish this magazine.

Henrietta Gendler (Upper Sixth, Queens), the president of Gaussian Group, for her support of the maths events and society.

Mr Yadsan for designing the maths magazine.

“The mathematician needs no laboratories or supplies. A piece of paper, a pencil, and creative powers form the foundation of his work”.

Alexander Khinchin.



# GAUSSIAN GROUP

2018 -2019

Gaussian Group members held several meetings this year.

Gaussian Group meetings started with *the Random Number Generators* this year.

On the philosophical side, we explored the followings:

What is randomness? Is it lack of causal relations? Is it a property of mind or is it a property of the universe? If it is a property of mind then if were intelligent enough, could we see a pattern, for example in the decimal expansion of Pi? If randomness is a property of the universe that is randomness exists in the absence of human mind, then whole mathematical physics is mere illusion. That is human mind connects unconnected events to make sense of it.

Can we sensibly talk about one thing being only a property of mind or solely being property of the universe? Are we not a part of the universe? Heisenberg tried to separate quantum from classical, Heisenberg's cut. Can we separate our mind from the universe?

On the mathematical side we explored and investigated normal and random numbers.

A number is said to be normal to base b if its base-b expansion has each digit appearing with average frequency tending to b^(-1), e.g. 0,1,2…,9. Prob is 1/10 in base 10. Intuitively this means that no digit, or (finite) combination of digits, occurs more frequently than any other, and this is true whether the number is written in base 10, binary, or any other base. A normal number can be thought of as an infinite sequence of coin flips (binary) or rolls of a die (base 6).

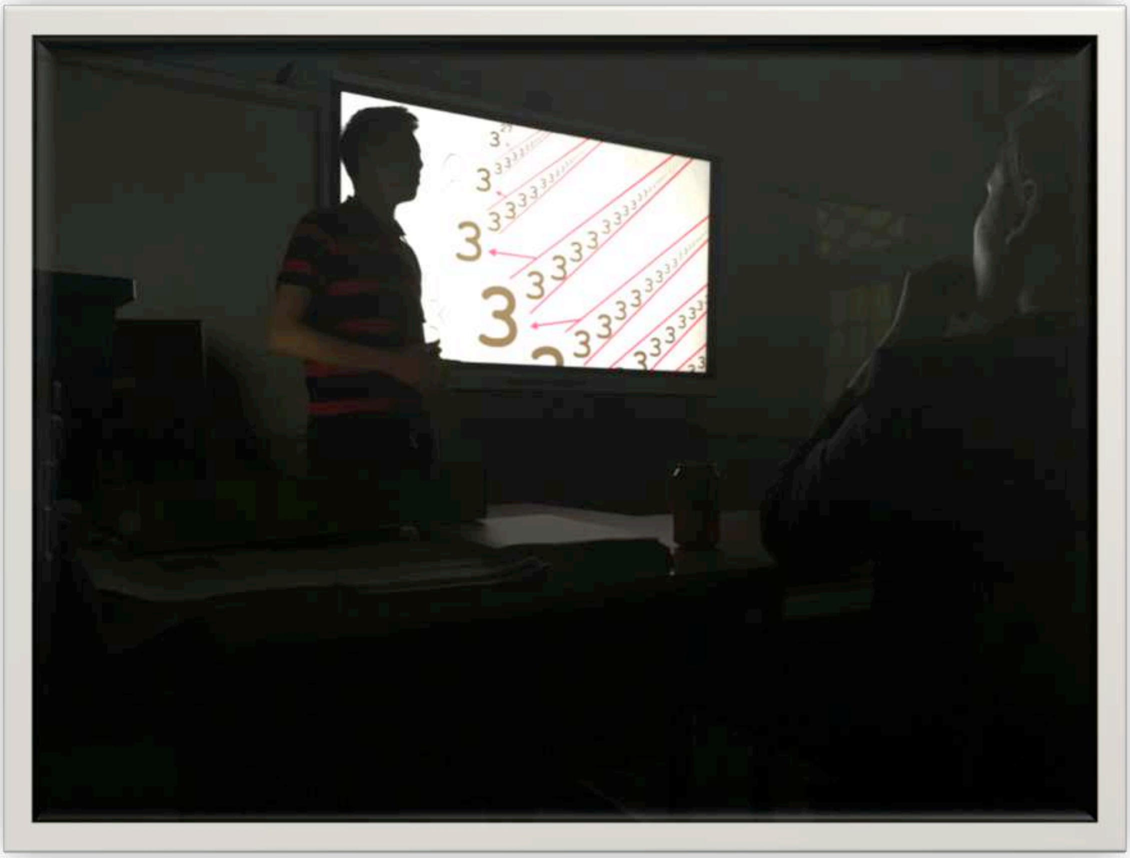
It is widely believed that the (computable) numbers √2, π, and e are normal, but a proof remains elusive.

This definition turns out to be counter intuitive. For example, ½ is not a normal number because ½ = 0.5000000000... we are certain that the next digit will be zero. 0.238526262626… is also not a normal number.

An example of normal numbers would be
0.123456789 10 11 12 13 14 … normal in base 10

It is widely believed that Pi is a normal number.

This talk was concluded by investigating how quantum mechanics uses radioactive decay and Eigenstate of the wave function to generate random numbers.



**Student Presentations**

There were several student presentations this year.

Charles Xu (Upper Sixth, Bruce): Pascal's Triangle and connecting different branches of mathematics.

Jack Boswell "The Black Scholes Model" strong connection between mathematics and economics.

Special thanks to Mr Stanworth for his very interesting talk on '**Can Chimpanzees Do Algebra?'**

In his talk, Mr Stanworth connected mathematics with experimental psychology. What do studies with primates teach us about mathematical cognition, and the capabilities of our own brains? We'll look at how research methods in psychology are developing our understanding of how we process mathematics in our own grey matter.


Rufus Esdale on Gambling and Cameron on the History of the Universe.

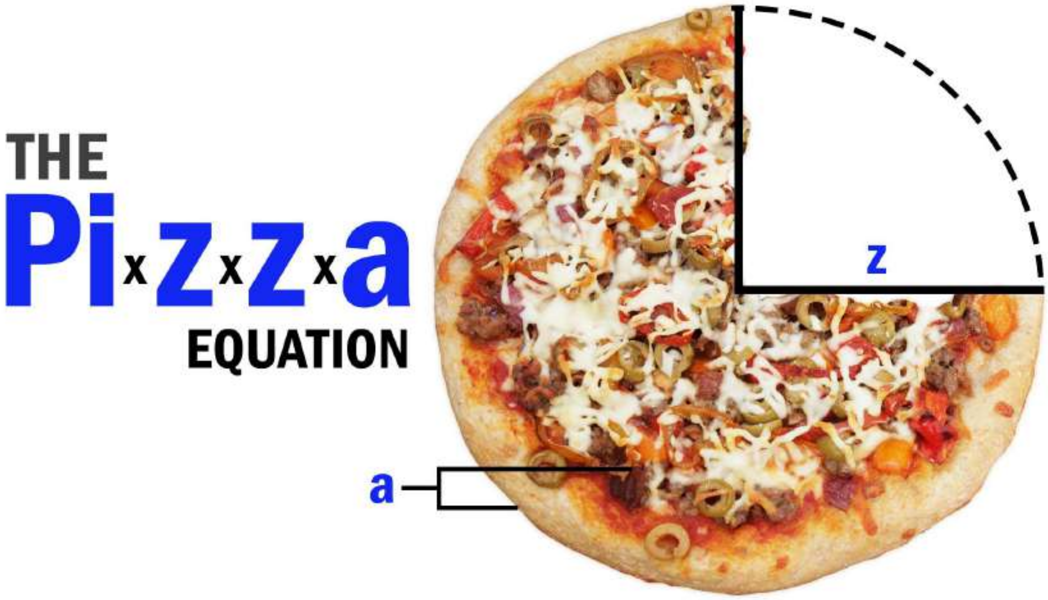
Andy Wu (Upper Sixth, Grenville) gave a very interesting and thought provoking talk on Probability and Randomness. When a fair, six sided die is thrown, P(5 or any other number) = 1/6. But when die is thrown twice, P(5,5) significantly differs from P(5). Andy questioned whether objects have memory.

Rohan Sekhri and Poom Narongpun (Upper Sixth Walpole), presented Bayesian Statistics.

Wei Lang Zhao (Upper Sixth, Grenville) on Large numbers.

We thank all our speakers for their time, effort and interesting talks.

**Movie and Pizza Night**



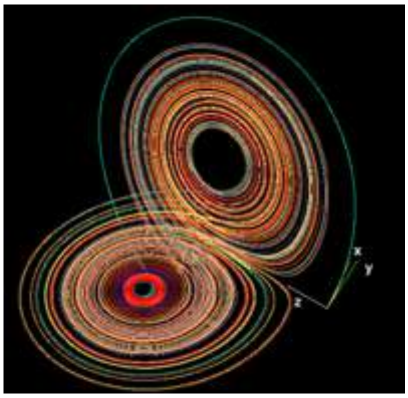

Gaussian Group's last event was Pizza & Movie night this academic year. Member enjoyed and celebrated the end of year by watching Good Will Hunting.

Special thank you to Henrietta Gendler for organising this event.

# Predator-Prey Modelling

## Mathematical Modelling

Mathematical models are descriptions of a system using mathematical concepts and language. Seemingly simple and mundane in the A-Level curriculum, true mathematical modelling entails more than meets the eye, with every system being very different in terms of its properties - many characteristics must be considered, such as linearity and non-linearity, static and dynamic behaviour and whether randomness is present (whether it is a deterministic or stochastic system).



(The Lorenz attractor, developed in 1963 to model atmospheric convection – an example of a chaotic system)

With advancements in technology seemingly arising more and more, the field of mathematical modelling has become its own discipline, with it being used to extend human knowledge of almost all fields, whether it be predicting where riots will occur in South America, studying the behaviour of black holes, or even observing the behaviour of predator-prey cycles. Modelling the most seemingly ordinary natural phenomena such as bird flocking can be used to develop models for fluid dynamics, and it is this transferability that makes any system worth modelling – even ecological ones, where one can use a model to analyse populations rather than throwing quadrats around to sample daisy populations.

## The Logistical Curve

Between 1838 and 1847, Pierre François Verhulst introduced his “logistical curve”, using it as a model of population growth. The initial stage of growth is roughly exponential (geometric), then as saturation begins, the growth slows to a more linear (arithmetic) rate, before reaching the maximum population for the given system.

This function simply results in an S-shaped curve, where certain parameters can be altered for different systems :

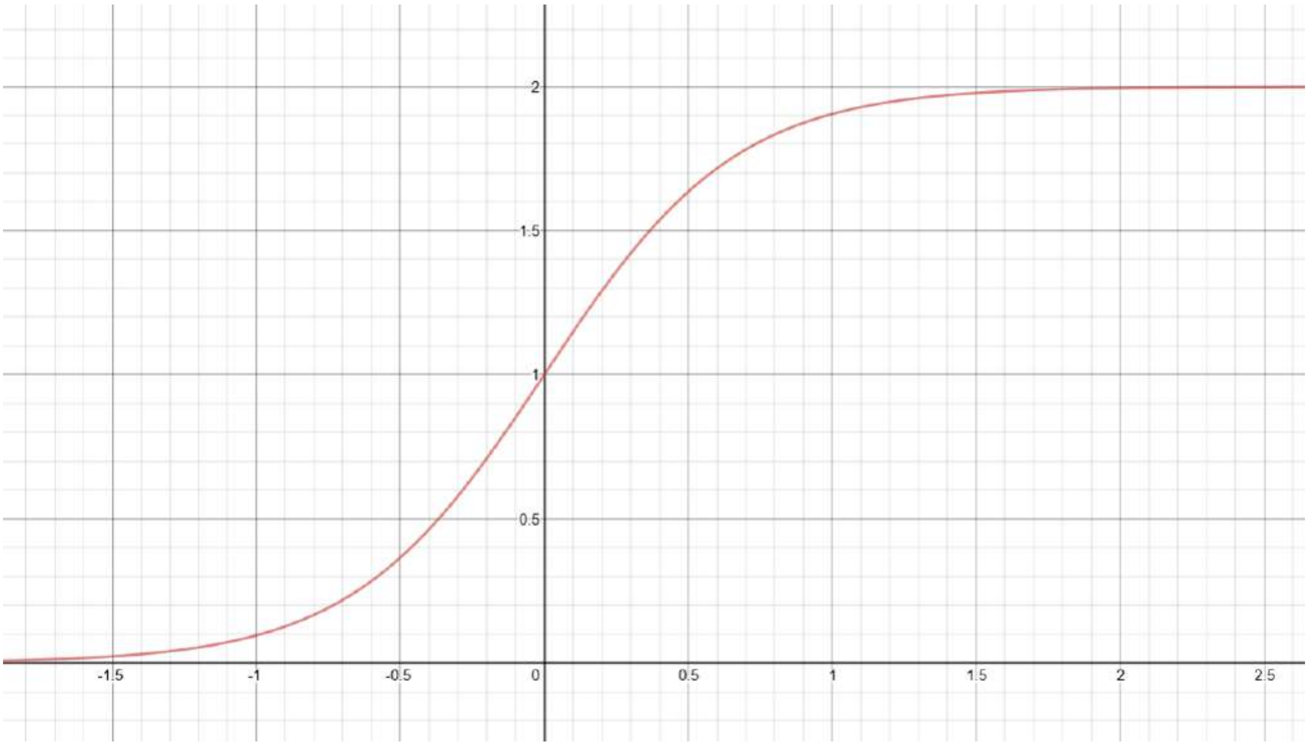
$$\boldsymbol{f(x) = \frac{L}{1 + e^{-k(x - x_0)}}}$$

*e* - the natural logarithm base

$x_0$ – the x-value of the sigmoid's midpoint

L – the curve's maximum value

k – the logistic growth rate/ gradient




(The above sigmoid is produced when L = 2, $x_0$= 0, and k = 3)

This model is used in chemistry to look at changes in reactants and products, medicine to model the growth of tumours, and even in linguistics to predict language change. In terms of its suitability of modelling populations, it's suitable for modelling populations in the process of uninterrupted growth (such as the human population, where the death rate hasn't overtaken birth rate), but not so much for those whose death rate overtakes the birth rate temporarily, due to factors such as predation, changes in food availability, or potentially abrupt climate changes.

## The Lotka-Volterra Equations

In 1910, Polish-American mathematician Alfred Lotka published a paper on autocatalytic chemical reactions, more or less using the logistical curve as a model. Later on, however, in 1920, Lotka extended Verhulst's model with the help of Soviet mathematician Andrey Kolmogorov into organic systems, using a plant and herbivorous animal species as an example. In 1925, he then published his equations in a book on biomathematics, which was closely followed by Italian mathematician Vito Volterra in 1926, who had synthesised the same equations to analyse fish populations in the Adriatic Sea to explain the findings of marine biologist Umberto D'Ancona, who was studying percentages of predatory fish caught during the years of World War I.

The Lotka-Volterra equations are a pair of first-order nonlinear differential equations, being used to describe the dynamics of populations in a biological system:

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = \delta xy - \gamma y$$

$x$ - the number of prey

$y$ - the number of predators

$\frac{dx}{dt}$ and $\frac{\mathrm{dy}}{\mathrm{dt}}$ – the instantaneous growth rates of the two populations

$\alpha, \beta, \gamma, \delta$ - positive, real parameters that are adjusted based on the type of interaction between the two populations

As a massively simplified model, the Lotka-Volterra equation makes a number of assumptions, not necessarily noticable in nature, about the environments and evolution of the predator and prey populations:

1) The prey population finds ample food at all times.
2) The food supply of the predator population is entirely dependant upon the prey population.
3) The rate of change of population is proportional to its size.

4) During the process, the environment does not change in favour of one species, and genetic adaptation has no effect.
5) Predators have limitless appetites.
6) The system is deterministic (no randomness involved) and continuous.

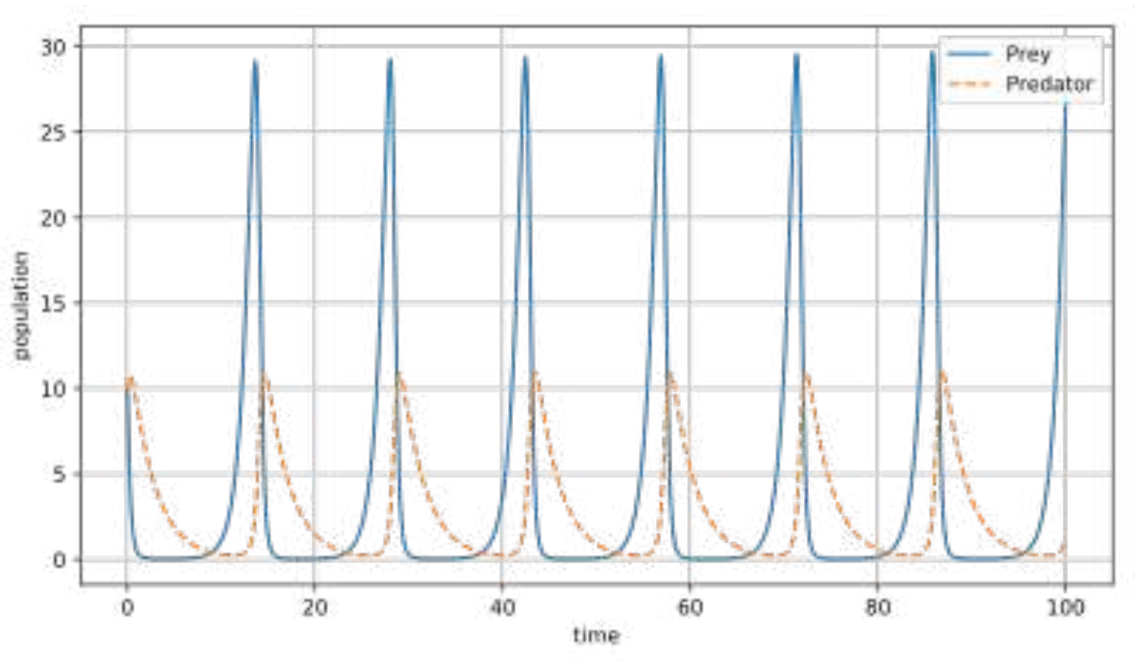
In terms of a physical meaning of the equations, it is fairly easy to understand:

$$\frac{dx}{dt} = \alpha x - \beta xy$$

The prey are assumed to have an unlimited food supply and to reproduce exponentially, unless subject to predation. The exponential growth is represented by the term $\alpha x$, and the rate of predation is proportional to the rate at which prey and predator meet, represented by $\beta xy$, so that if either are 0, then there is no predation.

$$\frac{dy}{dt} = \delta xy - \gamma y$$

In the predator equation, $\delta xy$ represents the growth of the predator population (similar to the $xy$ term in the prey equation, although with a different constant as they are not necessarily equal). $\gamma y$ accounts for the loss of predators due to either natural death or emigration, leading to an exponential decay in the absence of prey.




(An example of a predator-prey cycle for baboons (prey) and cheetahs (predator))

As shown above, the Lotka-Volterra model produces a nice representation of the relationship between prey and predator – the idea of a cycle. This is great in that it can be used as a basis for development of more specific prey-predator models in different contexts, but it doesn't allow any inferences to be made about how other biotic and abiotic factors such as climate will affect the population cycles.

If the model were to be adjusted to include more variables, the system would become a chaotic one, which is extremely hard to model. Chaotic systems are not actually random at all, even if they do exhibit seemingly random behaviour.

The key property of a chaotic system is that it is deterministic. A term coined by Laplace, determinism (more specifically classical determinism) is the concept of a clockwork universe, where all the most underlying and fundamental laws of the universe are known, so that – if enough is known about a system or an element – its entire history and future could be predicted with pinpoint accuracy. The uncertainty in chaotic systems inherently arises from the unfortunate prospect of us not having one unified theory for the entire universe, and not having infinite accuracy when measuring variables in a system. Infinitely small changes in initial conditions would not be observed by humans, but would result in a completely different outcome over time, although we would observe both inputs to be the same. It is true indeed that the exact same inputs would result in the same outputs, but we merely cannot achieve or recognise when conditions are the exact same.

## The Takeaway Lesson

Although what I have just said may seem to completely ruin the concept of modelling, because we can never have a perfect model, it is important to note that modelling is what all true sciences are. One clear example is the model of the atom, which has undergone drastic change since the early days of the Ancient Greek philosophers, and even today there are intense debates over what an electron even is! We will never have pinpoint accuracy, and that is simply due to the fact that we cannot comprehend infinity, but we can continue to learn, observe and appreciate the unknown, and get a decent bite of the constantly growing cherry that is human knowledge.

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**The Man of Principles**

Grigori Perelman was awarded $1m for proving one of the most famous open questions in maths, the Poincaré Conjecture. But the Russian recluse has refused to accept the cash. He had already turned down maths' most prestigious honour, the Fields Medal in 2006. "*If the proof is correct then no other recognition is needed,*" he reportedly said. The Poincaré Conjecture was first stated in 1904 by Henri Poincaré and concerns the behaviour of shapes in three dimensions. Perelman is currently unemployed and lives a frugal life with his mother in St Petersburg.

Just two years later, in November 2002, a Russian mathematician posted his proof of the Poincare Conjecture on the Internet. He was not the first person to claim he'd solved the Poincare-he was not even the only Russian to post a putative proof of the conjecture on the Internet that year-but his proof turned out to be right.

And then things did not go according to plan-not the Clay Institute's plan or any other plan that might have struck a mathematician as reasonable. Grigory Perelman, the Russian, did not publish his work in a refereed journal. He did not agree to vet or even to review the explications of his proof written by others. He refused numerous job offers from the world's best universities. He refused to accept the Fields Medal, mathematics' highest honor, which would have been awarded to him in 2006. And then he essentially withdrew from not only the world's mathematical conversation but also most of his fellow humans' conversation.

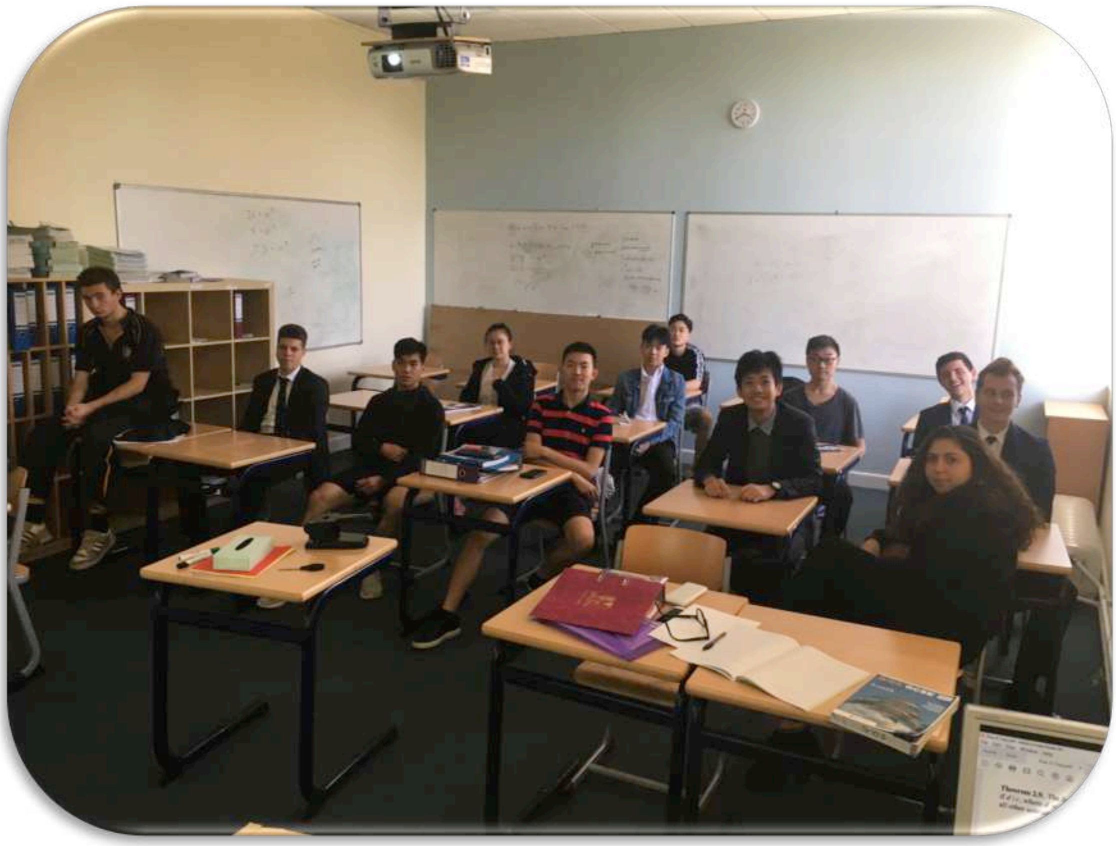
His objection to the Fields Medal, though never stated as clearly, seemed to have been twofold: first, he no longer considered himself a mathematician and hence could not accept a prize intended for the encouragement of midcareer researchers; and second, he wanted no part of the 1CM, with all the attendant publicity, speeches, ceremony, and king of Spain.

Schlicter said, "Go down deep enough into anything, and you will find mathematics."

This year we offered many academic activities in the maths department, and every activity is an opportunity for us to expand your intellectual capacity. These activities are for Stoics who desire to understand mathematics at a deeper level.

## VERTICAL STRETCH

Vertical Stretch sessions take place once a week. From September to January, we studied methods of solving linear and nonlinear integer equations (Number theory). From January to May (this is to be discussed) we studied a course advanced Calculus and more.



## NUMBER THEORY

Carl Friedrich Gauss said, "Mathematics is the queen of the sciences and number theory is the queen of mathematics". In this course, we started with the fundamental concept of gcd (greatest common divisor) and built towards advanced level theorems such as Euclid's algorithm, and general solution to linear Diophantine Equations. During this course, we have seen how a basic concept, such as prime decomposition is linked to the Fundamental Theorem of arithmetic. We have also scratched the surface of unique and non-unique factorisation domain. We have continued to work further on integer equations.

We have learnt how to use the modular arithmetic to solve linear simultaneous integer equations. We have also studied Euler Totient function and Fermat's little theorem in this course.

## CALCULUS

We started to this course by giving the formal definition (epsilon-delta) of continuity and limit, including left-hand and right-hand limit. After that, we studied the laws of limit and how to apply these laws to find the limit (given that it exists) of rational functions.

We have linked the idea of limits and continuity to the derivatives and studied the derivative of a function at a point. Thereon we embarked on to the numerical integration building towards the idea of the infinitesimal sum. We further studied Riemann summable functions.
The highlight of this course has been the study of the Fundamental Theorem of Calculus (F.T.C.). We have carefully studied how the area is linked to the tangent? That is, we studied integrals and derivative and how F.T.C. connect the two. We have finished this course by studying double integrals (rectangular and over the general region) in cartesian and polar coordinate forms primarily focussing on Fubini's theorem and its consequences.

Final weeks of this academic term, we studied the mathematics of quantum mechanics. In this short introductory course, we have covered the basics of statistical interpretation by starting the definition of the modulus square of the wavefunction. We further studied how to normalise it and the physical meaning of normalising the wave function. Using probability density interpretation, we learnt how to calculate the expectation values. We used Fubini's theorem to work out the area of bell-shaped curves. Using Schrodinger's equation, we have proved that once the wave function is normalised, it remains normalised.

We have finished this course by calculation expectation values for position and momentum, then testing Heisenberg's uncertainty principle.

# LOCAL MAXIMUM

“Mathematics is the door and key to the sciences”. — Roger Bacon

In these weekly running sessions, we focused on problem-solving skills. This is a preparation for Stoics who are planning to study maths or mathematical sciences at the top end universities. We tackle MAT, Step I, II, III and world maths Olympiad questions.

# MATHS PROJECT

To spread the love of rationality, to inspire and encourage the student and to advertise the most sophisticated and disciplined way of thinking, we have covered challenging topics in this year's maths project. We started by convergence and continuity of finite and infinite sequences and series this year. After formally defining convergence and divergence, we studied the strict condition of absolute convergence and he under which circumstances infinite sequences commute. We have further studied Integral tests, Comparison and limit comparison test, Ratio test, root test and alternating series test. Thereon we embarked on power series focusing on Taylor's Theorem. Taylor's expansion of multivariable functions and using Hessian Matrix to work out the nature of the stationary points. We used this to prove a historical problem called the Basel Problem.

In the second part of the project, we studied convex functions. We started off being the definition of convexity and proved Hermite-Hadamard Inequality using this definition. We have then determined Jensen's and Young's inequality. Using these result, we proved and looked at some applications of Hölder's inequality.

# Applications of Taylor Series

**This article discusses three important applications of Taylor series:**

*1. Using Taylor series to find the sum of a series.*

*2. Using Taylor series to evaluate limits.*

*3. Using Taylor polynomials to approximate functions.*

## Evaluating Infinite Series：

It is possible to use Taylor series to find the sums of many different infinite series. The following examples illustrate this idea.

EXMAPLE 1  Find the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

SOLUTION  The Taylor series for e^x

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = e^x.$$

The sum of the given series can be obtained by substituting in x=1:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e.$$

In the above example, note that we get a different series for every value of x that we plug in. For example,

$$1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots = e^2.$$

and

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1} = \frac{1}{e}.$$

## Limits Using Power Series

When taking a limit as x→0 , you can often simplify things by substituting in a power series that you know.

EXAMPLE 3 Evaluate $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

SOLUTION We simply plug in the Taylor series for sin (x):

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x}{x^3} &= \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x}{x^3} \\
&= \lim_{x\to 0} \frac{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^3} \\
&= \lim_{x\to 0} -\frac{1}{3!} + \frac{1}{5!}x^2 - \frac{1}{7!}x^4 + \cdots = -\frac{1}{3!} = -\frac{1}{6}
\end{aligned}$$

# Taylor Polynomials

A partial sum of a Taylor series is called a Taylor polynomial. For example, the Taylor polynomials for are e ^x:

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= 1 + x \\
T_2(x) &= 1 + x + \frac{1}{2}x^2 \\
T_3(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \\
&\vdots
\end{aligned}$$

You can approximate any function by its Taylor polynomial:

$$f(x) \approx T_n(x)$$

If you use the Taylor polynomial cantered at a, then + the approximation will be particularly good near x=a.

Let $f(x)$ be a function. The **Taylor polynomials** for $f(x)$ centered at $x = a$ are:

$$\begin{aligned} T_0(x) &= f(a) \\ T_1(x) &= f(a) + f'(a)(x-a) \\ T_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ &\vdots \end{aligned}$$

You can approximate $f(x)$ using a Taylor polynomial.

1st-degree Taylor polynomial is just the tangent line to f(x) at x=a:

This is often called the linear approximation to near x=a, i.e. the tangent line to the graph. Taylor polynomials can be viewed as a generalization of linear approximations. In particular, the 2nd-degree Taylor polynomial is sometimes called the quadratic approximation, the 3rd-degree Taylor polynomial is the cubic approximation and so on

# MATHS JOKES ☺

What is conditional probability?

Maybe, maybe not

Change 7/8 to a decimal.

7.8

Simplify the following equation.

$$\frac{\sqrt{5}}{5}$$

$$\frac{\sqrt{\cancel{5}}}{\cancel{5}} = \sqrt{\phantom{5}}$$

Why are mathematicians afraid to drive a car?

Because the width of the road is negligible compared to its length.

Why didn't Newton discover group theory?

Because he wasn't Abel.

Have you heard the latest statistics joke?

Probably…

How do you save a drowning statistician?

Stop holding his head underwater.

What did the statistics teacher say to console his failing student?

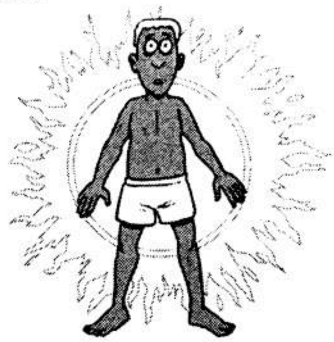
"Look on the bright side—you're in the top 90% of the class!"

Where do they put mathematicians who commit crimes?

Prism... so they can be with other convex.

What do circles and beaches have in common?

Both have tan gents.



$$\int 3(ice)^2 d(ice) =$$ 

A biologist, a physicist and a mathematician were sitting at a street cafe. Across the street, a man and a woman entered a building. A few minutes later, the same man and woman exited the building with another person.

"Ah," says the biologist. "They have multiplied!"

"No," sighs the physicist. "It is an error in measurement."

The mathematician says, "If exactly one person enters now, the building will be empty."